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Density of states of sparse random matrices

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Abstract. A supersymmetric formalism is used to derive a set of equations giving the density of states of any real, symmetric, sparse random matrix as a function of the distribution of non-zero elements and the mean number of non-zero elements per row, p. In the matrix where the non-zero elements take the values ± 1 with equal probability the equations are solved as $p \to \infty$ recovering results obtained previously with the replica method. As $E \to 0$ the density of states $\rho(E)$ behaves as $1/E(\ln(E))^2$. The more general case where ± 1 occur with unequal probabilities is also considered.

1. Introduction

In this paper we use anticommuting variables to derive a set of equations which determine the eigenvalue spectrum of a set of large, sparse random matrices. These matrices have a mean finite number p of non-zero elements per row and a general distribution $\rho(J)$ of non-zero elements. Although the eigenvalue spectra of large random matrices have been studied for some time (e.g. Mehta 1967) there are very few results for matrices which are sparse.

Matrices such as these arise in the study of mean field spin systems in which the exchange interactions are very dilute, but of infinite range, so that the average connectivity is finite. These models fall into two classes: random lattices with either average finite connectivity (Viana-Bray models, see Viana and Bray (1985), Mezard and Parisi (1987), Kanter and Sompolinsky (1987)) or fixed connectivity. The latter class is related to that of a spin system on a Bethe lattice, although care must be taken with the boundary conditions on a Bethe lattice due to the finite fraction of spins on the surface (see for instance Carlson *et al* 1988, Lai and Goldschmidt 1989). These systems are closely connected with combinatoric optimisation problems such as colouring, matching, graph partitioning and the travelling salesman problems. In this paper we consider the first class of models with finite mean connectivity only.

In a previous study (Rodgers and Bray 1988) the replica method was used to evaluate the eigenvalue spectrum of a matrix with p mean, non-zero elements per row. These non-zero elements took the values ± 1 with equal probability. A nonlinear integral functional equation was found, the solution of which led to the density of states. For $p \to \infty$ a generalised Wigner semicircle was obtained and for finite p the Lifshitz tails extending beyond the semicircle were calculated.

This study was motivated by a desire to understand the role which Griffiths singularities (Griffiths 1969) played in the statics and dynamics of dilute spin systems in the Griffiths phase (i.e. at a temperature between the transition temperatures of the dilute and non-dilute systems). This is a question which has received much recent interest (e.g. Dhar 1983, Randeria *et al* 1985, Bray 1987, Bray and Rodgers 1988). It has been argued (Bray and Moore 1982, Hertz *et al* 1979) that the Griffiths phase is controlled by the spectrum of the inverse of the susceptibility matrix. However determination of the inverse of the susceptibility matrix is a highly nonlinear problem so as a first step the exchange matrix was considered. These two matrices are linearly related at high temperature.

In this work we use anticommuting variables in an attempt to repeat and extend the same calculation. This approach allows us to examine the problem of replica symmetry breaking and also to consider a larger class of matrices where the non-zero elements take a general distribution $\rho(J)$. The result can be formulated as a set of three nonlinear integral functional equations, the solution of which leads to the density of states. Inserting the symmetric distribution $\rho(J)=[\delta(J+1)+\delta(J-1)]/2$ leaves three non-trivial equations which we cannot reduce to the one obtained from the replica method. However, to order $1/p$ in a large p expansion, the two methods reveal the same results. The case of a non-symmetric distribution is also examined.

2. Eigenvalue spectrum

A real symmetric $N\times N$ matrix $\boldsymbol{J}$ is considered where the elements are independently distributed with a probability distribution

$$P(J_{ij})=(1-p/N)\delta(J_{ij})+(p/N)\rho(J_{ij}). \tag{1}$$

Hence p is the mean number of non-zero elements per row and we assume $\rho(J_{ij})$ is normalised and has no delta function at $J_{ij}=0$.

The normalised average density of states $\rho(\mu)$ for a particular realisation of the disorder $\{J_{ij}\}$ is given by the Green function

$$G(\mu)=\frac{1}{N}\sum_i\left\langle i\left|\frac{1}{\mu I-J}\right|i\right\rangle \tag{2}$$

with

$$\rho(\mu)=1/\pi\ \mathrm{Im}\ G(\mu+\mathrm{i}\varepsilon) \tag{3}$$

where $\varepsilon>0$ is infinitesimal.

To evaluate $\rho(\mu)$ we introduce the generating function

$$Z(\mu)=\int\prod_i\left(\frac{\mathrm{d}\phi_i\,\mathrm{d}\phi_i^*}{\pi}\right)\exp\left\{\sum_{ij}\phi_i^*(\mu\delta_{ij}-J_{ij})\phi_j\right\} \tag{4}$$

so that

$$G(\mu)=\frac{1}{N}\frac{\partial}{\partial\mu}\ln Z(\mu). \tag{5}$$

We will evaluate $G(\mu)$ and average over $\{J_{ij}\}$ by introducing Grassmannian variables via Berezin's formula (Berezin 1967, Efetov 1983)

$$\int\prod_i(\mathrm{d}\eta_i\,\mathrm{d}\eta_i^*)\exp\left\{-\sum_{ij}\eta_i^*a_{ij}\eta_j\right\}=\left[\int\prod_i\left(\frac{\mathrm{d}\phi_i\,\mathrm{d}\phi_i^*}{\pi}\right)\exp\left\{-\sum_{ij}\phi_i^*a_{ij}\phi_j\right\}\right]^{-1} \tag{6}$$

where $\{\eta_i\}$ are anticommuting variables. These variables satisfy Grassmannian integration rules, namely

$$\int \mathrm{d}\eta\, \mathrm{d}\eta^*(1, \eta, \eta^*, \eta\eta^*) = (0, 0, 0, 1) \tag{7}$$

and the commutation relations

$$\eta_i\eta_j + \eta_j\eta_i = 0 \qquad \eta_i\eta_j^* + \eta_j^*\eta_i = 0 \qquad \eta_i^*\eta_j^* + \eta_j^*\eta_i^* = 0. \tag{8}$$

If the differentiation in (5) is formally performed and the denominator rewritten in terms of $\{\eta_i\}$ we obtain

$$G(\mu) = \int \mathrm{D}\phi\, \mathrm{D}\eta\, \exp\bigg\{\mu \sum_i (\phi_i^*\phi_i + \eta_i^*\eta_i) - \sum_{(ij)} J_{ij}(\phi_i^*\phi_j + \eta_i^*\eta_j + \phi_j^*\phi_i + \eta_j^*\eta_i)\bigg\} \frac{1}{N}\sum_i \phi_i^*\phi_i \tag{9}$$

where $\mathrm{D}x = \Pi_i(\mathrm{d}x_i\, \mathrm{d}x_i^*)$ and we have used the fact that $\boldsymbol{J}$ is symmetric.

This expression can be simplified by introducing the superfields $\{\Phi_i(\theta)\}$ (see Efetov 1983) and the anticommuting variables θ and θ^* via

$$\Phi_i(\theta) = \phi_i + \eta_i^*\theta + \theta^*\eta_i + \phi_i^*\theta\theta^* \tag{10}$$

where θ and θ^* are independent of $\{\phi_i, \phi_i^*\}$ and $\{\eta_i, \eta_i^*\}$. Thus the integrals of $\{\Phi_i(\theta)\}$ over θ and θ^* give

$$\int \mathrm{d}\theta\, \mathrm{d}\theta^*\, \Phi_i(\theta)\Phi_j(\theta) = \phi_i^*\phi_j + \phi_j^*\phi_i + \eta_i^*\eta_j + \eta_j^*\eta_i \tag{11}$$

and (9) becomes

$$G(\mu) = \int \mathrm{D}\phi\, \mathrm{D}\eta\, \exp\bigg\{\int \mathrm{d}\theta\, \mathrm{d}\theta^*\bigg(\frac{\mu}{2}\sum_i \Phi_i^2(\theta) - \sum_{(i,j)} J_{ij}\Phi_i(\theta)\Phi_j(\theta)\bigg)\bigg\} \frac{1}{N}\sum_i \phi_i^*\phi_i. \tag{12}$$

Averaging over the probability distribution of $\{J_{ij}\}$ (1) and dropping the subextensive terms gives

$$G(\mu) = \int \mathrm{D}\phi\, \mathrm{D}\eta\, \exp\bigg\{\int \mathrm{d}\theta\, \mathrm{d}\theta^* \frac{\mu}{2}\sum_i \Phi_i^2(\theta) + \frac{p}{N}\sum_r b_r \sum_{ij}\bigg(\int \mathrm{d}\theta\, \mathrm{d}\theta^*\, \Phi_i(\theta)\Phi_j(\theta)\bigg)^r\bigg\} \frac{1}{N}\sum_i \phi_i^*\phi_i \tag{13}$$

where

$$b_r = \frac{(-1)^r}{r!}\int_{-\infty}^{\infty} \rho(J)J^r\, \mathrm{d}J. \tag{14}$$

As the superfields $\Phi_i(\theta)$ are commuting variables the identity

$$\sum_{ij}\bigg(\int \mathrm{d}\theta\, \mathrm{d}\theta^*\, \Phi_i(\theta)\Phi_j(\theta)\bigg)^r = \int \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_1\, \mathrm{d}\theta_1^* \ldots \mathrm{d}\theta_r^*\bigg(\sum_i \Phi_i(\theta_1)\ldots\Phi_i(\theta_r)\bigg)^2 \tag{15}$$

holds and we can rewrite $G(\mu)$ as

$$G(\mu) = \int \prod_r (\mathrm{d}y^{(r)}_{\theta_1\ldots\theta_r}) \exp\bigg\{-\frac{Np}{2}\sum_r b_r \int (y^{(r)}_{\theta_1\ldots\theta_r})^2\, \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_r\, \mathrm{d}\theta_1^* \ldots \mathrm{d}\theta_r^* + N \ln W\bigg\} \tag{16}$$

where

$$W=\int D\phi\, D\eta \exp\left\{p\sum_r b_r \int y^{(r)}_{\theta_1\ldots\theta_r}\Phi(\theta_1)\ldots\Phi(\theta_r)\,d\theta_1\ldots d\theta_r\, d\theta_1^*\ldots d\theta_r^* + \frac{\mu}{2}\int d\theta\, d\theta^*\, \Phi^2(\theta)\right\}. \tag{17}$$

The variables $\{y^{(r)}_{\{\theta\}}\}$ are given by the saddle point

$$y^{(r)}_{\theta_1\ldots\theta_r}=\langle\Phi(\theta_1)\ldots\Phi(\theta_r)\rangle \tag{18}$$

where $\langle\ldots\rangle$ represents a normalised average over the integrand W. If we introduce the function

$$F(\phi,\eta)=\sum_r b_r\int y^{(r)}_{\theta_1\ldots\theta_r}\Phi(\theta_1)\ldots\Phi(\theta_r)\,d\theta_1\ldots d\theta_r\, d\theta_1^*\ldots d\theta_r^* \tag{19}$$

then F satisfies the equation

$$F(\psi,\rho)=\int dJ\,\rho(J)\int d\phi\, d\eta \exp\{pF(\phi,\eta)+\mu(\phi^*\phi+\eta^*\eta) + J(\phi^*\psi+\psi^*\phi+\eta^*\rho+\rho^*\eta)\} \times\left\{\int d\phi\, d\eta \exp\{pF(\phi,\eta)+\mu(\phi^*\phi+\eta^*\eta)\}\right\}^{-1}. \tag{20}$$

Now $F(\psi,\rho)$ can always be parametrised as

$$F(\psi,\rho)=A(\psi,\psi^*)+B(\psi,\psi^*)\rho^*\rho \tag{21}$$

and if $A(\psi,\psi^*)$ has a coefficient A_1 on its $\psi\psi^*$ term and $\rho(J)=[\delta(J+1)+\delta(J-1)]/2$ then $G(\mu)=A_1$.

These equations are our main result and give the eigenvalue spectrum of any sparse random symmetric matrix with independent random elements.

3. Solutions

Equation (20) is difficult to solve in general and we restrict ourselves to the case where the non-zero elements take the values ±1 with equal probability. This is the problem which has already been considered with the replica method (Rodgers and Bray 1988); the eigenvalue spectrum in the large p limit was found to be given by

$$\rho(\mu)=\frac{2}{\pi\mu_c^2}(\mu_c^2-\mu^2)^{1/2}\left[1+\frac{1}{p}\left(1-\frac{4\mu^2}{\mu_c^2}\right)+O\left(\frac{1}{p^2}\right)\right] \qquad \text{for } \mu^2<\mu_c^2 \tag{22}$$

and $\rho(\mu)=0$ otherwise. μ_c^2 is given by

$$\mu_c^2=4\left[p+1+O\left(\frac{1}{p}\right)\right]. \tag{23}$$

This result can be obtained using the supersymmetric formalism. The first step is to notice that when $\rho(J)$ is symmetric the following gauge transformations hold:

$$\phi_i\Rightarrow e^{i\omega}\phi_i \qquad \phi_i^*\Rightarrow e^{-i\omega}\phi_i^* \qquad \eta_i\Rightarrow e^{i\omega}\eta_i \qquad \eta_i^*\Rightarrow e^{-i\omega}\eta_i^*. \tag{24}$$

Hence $A(\psi, \psi^*) = A(\psi\psi^*)$ and $B(\psi, \psi^*) = B(\psi\psi^*)$. Introducing variables $E = -\mu$, $u = E\phi\phi^*$ and $v = E\psi\psi^*$ into (20) gives

$$Z = \frac{1}{E}\int_0^\infty du \exp\left\{pA\left(\frac{u}{E}\right) - u\right\}\left[B\left(\frac{u}{E}\right) - E\right] \tag{25}$$

$$A\left(\frac{v}{E}\right)Z = \frac{1}{E}\int_0^\infty du \exp\left\{pA\left(\frac{u}{E}\right) - u\right\} \times\left[B\left(\frac{u}{E}\right) - E\right]\int I_0\left(2\frac{J}{E}\sqrt{(uv)}\right)\rho(J)\,dJ \tag{26}$$

$$B\left(\frac{v}{E}\right)Z = \frac{-p}{E}\int_0^\infty du \exp\left\{pA\left(\frac{u}{E}\right) - u\right\}\int I_0\left(2\frac{J}{E}\sqrt{(uv)}\right)J^2\rho(J)\,dJ \tag{27}$$

$$G(E)Z = \frac{1}{E^3}\int_0^\infty du\, u \exp\left\{pA\left(\frac{u}{E}\right) - u\right\}\left[B\left(\frac{u}{E}\right) - E\right]. \tag{28}$$

In the case where the non-zero elements only take the values ± 1 (28) may be replaced by

$$G(E) = A_1 \tag{29}$$

if $A(x) = 1 + A_1 x + O(x^2)$.

This set of equations is much more complicated than the equivalent set obtained from the replica method; there seems to be no simple relationship between them.

The large p expansion reveals the same results for $\rho(\mu)$; if we write

$$A(u) = 1 + \sum_{s=1}^\infty u^s \frac{A_s}{p^s} \qquad \text{where} \qquad A_s = \sum_{r=0}^s \frac{A_s^{(r)}}{p^r} \tag{30}$$

and

$$B(u) = \sum_{s=0}^\infty u^s \frac{B_s}{p^s} \qquad \text{where} \qquad B_s = \sum_{r=0}^s \frac{B_s^{(r)}}{p^r}$$

and expand (25)-(27) to order $1/p$ and eliminate B_0 and B_0^1 we obtain

$$A_1^0 = \frac{1}{E^2(1 - A_1^0)} \tag{31}$$

and

$$A_1^1 = \frac{1}{E^2}\left(\frac{A_1^1}{(1 - A_1^0)^2} + \frac{1}{E^4}\frac{1}{(1 - A_1^0)^5}\right). \tag{32}$$

These equations combined with (29) give $\rho(\mu)$ as obtained by the replica method (22) and (23).

The equations (25)-(27) can also be solved for $E \to 0$, giving the eigenvalue density in the centre of the spectrum. Making the change of variables $u' = u/E$ in (25)-(27) and expanding the Bessel functions for small E gives

$$A(v) \sim \frac{\tilde{A}}{v} \qquad \text{and} \qquad B(v) \sim \frac{\tilde{B}}{v} \tag{33}$$

where

$$\tilde{A} \sim \left(\frac{E}{\ln E}\right)^{1/2} \qquad \text{and} \qquad \tilde{B} \sim \left(\frac{\ln E}{E}\right)^{1/2} \tag{34}$$

and hence using (28) and (3) we get

$$\rho(E) \sim \frac{1}{E(\ln E)^2}. \tag{35}$$

We have also tried to solve equations (25–27) when $\rho(J)$ is non-symmetric, e.g. where $\rho(J) = \delta(J-1)$. This means that $A(\psi, \psi^*)$ and $B(\psi, \psi^*)$ can be functions of two independent variables, not one as before. We were, however, unable to find a self-consistent solution as $p \to \infty$. This suggests that $A(\psi, \psi^*) = A(\psi\psi^*)$ and $B(\psi, \psi^*) = B(\psi,\psi^*)$ for this non-symmetric distribution and hence that at least for $p \to \infty$ matrices with distribution

$$P(J) = \left(1 - \frac{p}{N}\right)\delta(J) + \frac{p}{N}[c\delta(J-1) + (1-c)\delta(J+1)] \tag{36}$$

where $0 \leqslant c \leqslant 1$, all have the same eigenvalue density as $p \to \infty$.

This seems plausible if one considers the matrix in question. It is very dilute, the non-zero elements are very rare, so in calculating the determinant the sign of the non-zero entry is determined at random by its position in the matrix. Thus the sign of the entry has no effect.

To conclude, we have used a supersymmetric formalism to calculate the density of states of a real symmetric sparse random matrix with an arbitrary distribution of non-zero entries. For the case where the non-zero entries took values ± 1 with equal probabilities, an explicit expression was obtained for the density of states as the mean number of non-zero entries per row went to infinity (22). This result coincided with a result obtained with the replica method assuming replica symmetry (Rodgers and Bray 1988) in the $p \to \infty$ limit. We also obtained an expression for the density of states in the centre of the spectrum (35). Finally we gave a heuristic argument which suggests that these results hold for the case of non-symmetric entries, i.e. when ± 1 occur with unequal probabilities.

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